Velocity and Acceleration in the Polar Basis

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1 Cartesian Coordinates Revisited

In 2D Cartesian coordinates, we can decompose the position vector $\mathbf{r}$ into a sum of its **coordinates** (x, y) (also called **components**) multiplying the **basis vectors** $\hat{\mathbf{x}}, \hat{\mathbf{y}}$:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}. \tag{1}$$

So how do we get the velocity vector? We take the *total* time derivative $\mathrm{d}/\mathrm{d}t$, which in Newton's notation is written with an overdot like $\dot{\mathbf{r}}$. We know that $\dot{\mathbf{r}}$ must have the components $(\dot{x}, \dot{y})$, but we will calculate it in a more general way by applying the product rule:

$$\dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{x}} + x\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{y}} + y\frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}t}. \tag{2}$$

Calculating the velocity vector this way may seem oddly pedantic, but it is illustrative of what is to come. In general, basis vectors depend on coordinates, a relationship we can write explicitly as a function $\hat{\mathbf{x}}(x, y)$. You don't usually notice this dependence because the Cartesian basis vectors are fixed and do not vary with coordinates; that is, the partial derivative of $\hat{\mathbf{x}}(x, y)$ with respect to either x or y is zero:

$$\frac{\partial\hat{\mathbf{x}}}{\partial x} = 0, \quad \frac{\partial\hat{\mathbf{x}}}{\partial y} = 0. \tag{3}$$

Thus, when we try to explicitly calculate $\mathrm{d}\hat{\mathbf{x}}/\mathrm{d}t$ using the multivariate chain rule[1], we will get zero:

$$\frac{\mathrm{d}\hat{\mathbf{x}}}{\mathrm{d}t} = \cancelto{0}{\frac{\partial\hat{\mathbf{x}}}{\partial x}}\frac{\mathrm{d}x}{\mathrm{d}t} + \cancelto{0}{\frac{\partial\hat{\mathbf{x}}}{\partial y}}\frac{\mathrm{d}y}{\mathrm{d}t} = 0, \tag{4}$$

and Equation (2) reduces to

$$\dot{\mathbf{r}} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{x}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{y}} = \dot{x}\,\hat{\mathbf{x}} + \dot{y}\,\hat{\mathbf{y}} \tag{5}$$

[1] Be very careful about which derivatives are partial and which ones are total in (4)!

as expected.

By the same argument, we can write the acceleration vector in terms of the Cartesian basis vectors as

$$\ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\hat{\mathbf{x}} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\hat{\mathbf{y}} = \ddot{x}\,\hat{\mathbf{x}} + \ddot{y}\,\hat{\mathbf{y}}. \tag{6}$$

2 The Polar Basis Vectors

Just as we define the basis vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ for the Cartesian coordinates (x, y), we can define $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ for the polar coordinates (r, θ). This time, however, the basis vectors *do* depend on the coordinates. Figure 1 shows the polar basis vectors for two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. You can imagine that the basis vectors are "attached" to the endpoints of the position vectors, with $\hat{\mathbf{r}}$ always pointing radially outwards and $\hat{\boldsymbol{\theta}}$ always pointing tangentially in the counterclockwise direction.

To translate the geometric definition of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ into algebra, we can write them in terms of the Cartesian basis vectors:

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}, \tag{7}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}. \tag{8}$$

The right triangles attached to $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ in Figure 2 illustrate this relationship.

Because of their dependence on θ, the total time derivatives of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ no longer vanish. With some foresight, we calculate them now.

What is $\dot{\hat{\mathbf{r}}} = \mathrm{d}\hat{\mathbf{r}}/\mathrm{d}t$? Applying the multivariate chain rule to (7), we have

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= \frac{\partial \cos\theta}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{x}} + \frac{\partial \sin\theta}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{y}} \\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\left(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}\right) \\ &= \dot{\theta}\,\hat{\boldsymbol{\theta}}.\end{aligned} \tag{9}$$

This makes sense geometrically: if we vary the coordinate by a small amount $\Delta\theta$, the resulting movement is circular, so $\hat{\mathbf{r}}$ must change in the tangential direction $\hat{\boldsymbol{\theta}}$ by $\Delta\theta$. See Figure 3 for an illustration of this geometric argument. Similarly, we can show that

$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\,\hat{\mathbf{r}}. \tag{10}$$

3 Velocity in the Polar Basis

Back to the problem of writing velocity and acceleration vectors in terms of the polar basis vectors. First, we must write the position vector $\mathbf{r}$ in terms of our new basis vectors. It turns out that $\mathbf{r}$ takes on a very simple form:

$$\begin{aligned}\mathbf{r} &= r\left(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\right) \\ &= r\,\hat{\mathbf{r}}.\end{aligned} \tag{11}$$

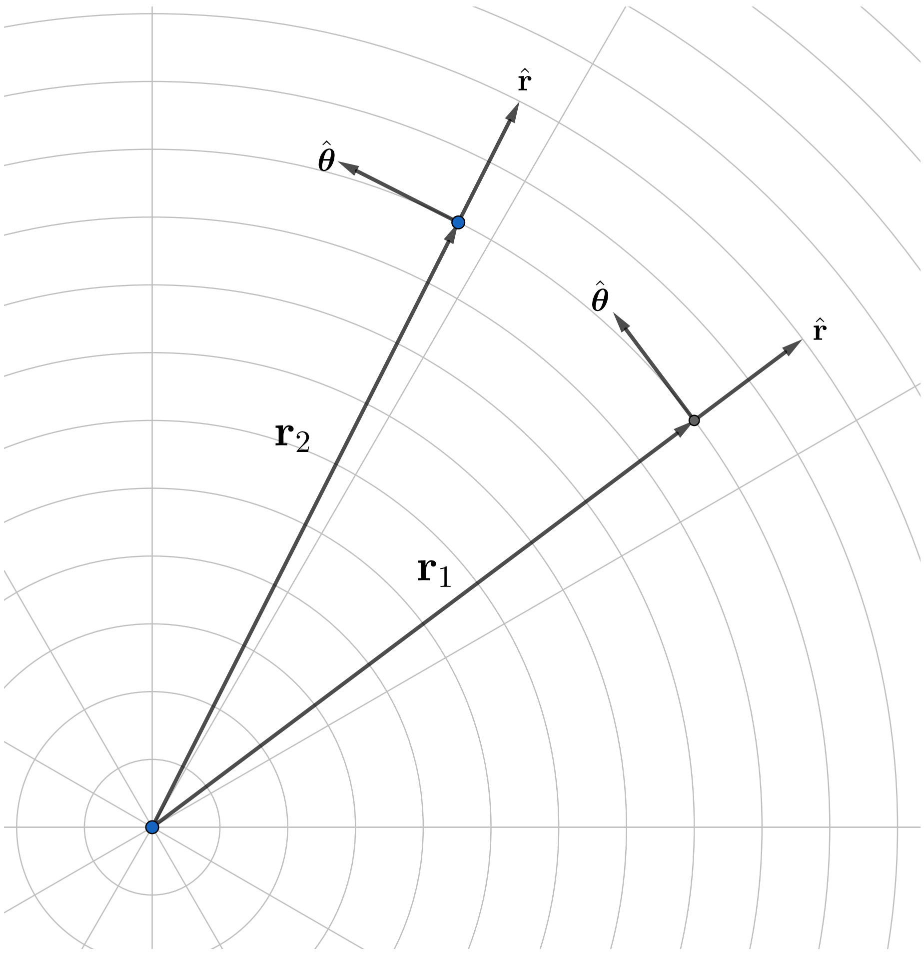

Figure 1: How the polar basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ vary as the position changes from $\mathbf{r}_1$ to $\mathbf{r}_2$.

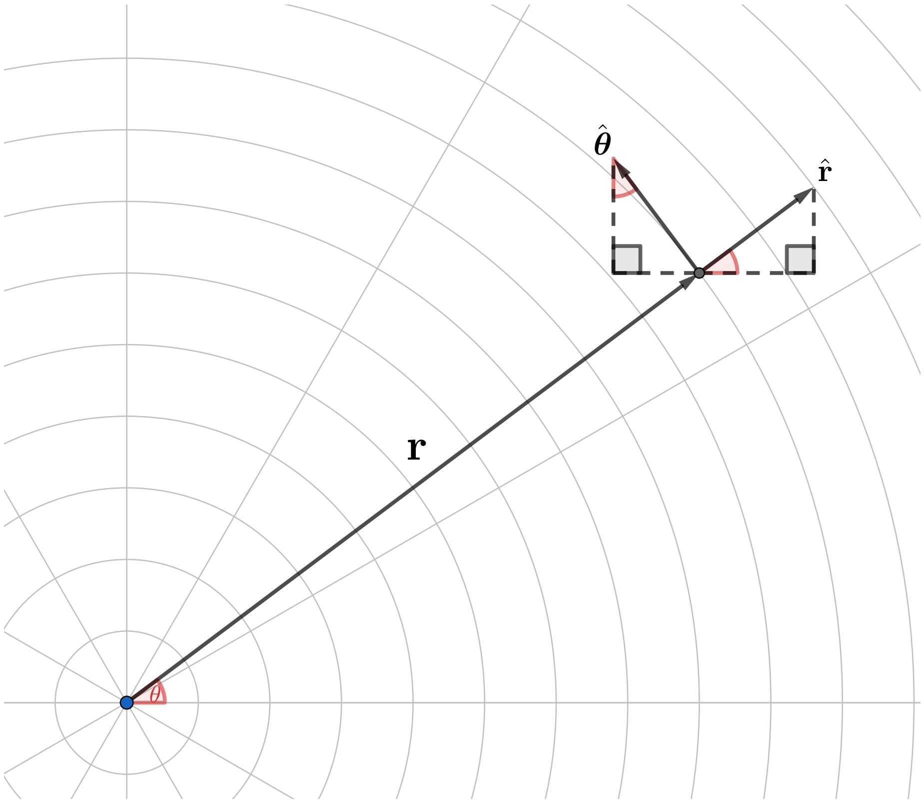

Figure 2: The dependence of polar basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ on θ (angle drawn in red).

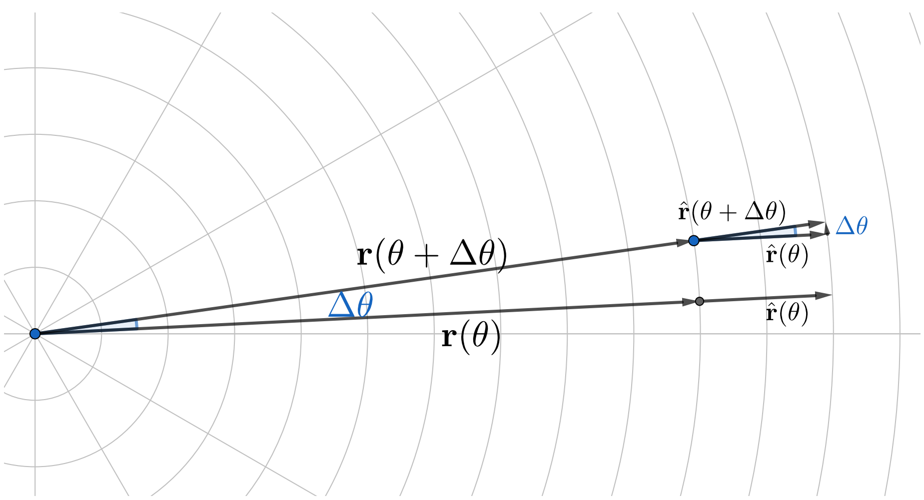

Figure 3: A geometric realization of $\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$.

You might be alarmed by the lack of θ in the last line, but never fear: $\hat{\mathbf{r}}$ already depends on θ. You can write this dependence explicitly as $\mathbf{r} = r\,\hat{\mathbf{r}}(\theta)$, but we rarely do so because it is always implied.

Emulating the procedure carried out in Equation (2) and referring to the expression for $\dot{\hat{\mathbf{r}}}$ in (9), we now take the total time derivative of $\mathbf{r}$:

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}} \\ &= \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}. \end{aligned} \tag{12}$$

The velocity vector is split into a radial component $\dot{r}$ and a tangential component $r\dot{\theta}$. You may recognize $r\dot{\theta}$ as simply the formula for converting angular speed to linear speed in circular motion.

4 Acceleration in the Polar Basis

We are now ready to tackle the final challenge: taking the total time derivative once more. In Newton's notation, we use the double overdot to write the second time derivative:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \ddot{\mathbf{r}}, \quad \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \ddot{r}, \quad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \ddot{\theta}, \quad \text{etc.} \tag{13}$$

Armed with the knowledge of $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$, however, all there is left to do is to apply the chain rule consistently. There are 5 items in (12), so we begin with 5 terms in our answer:

$$\ddot{\mathbf{r}} = \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\dot{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}\,\dot{\hat{\boldsymbol{\theta}}} \tag{14}$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\,\hat{\mathbf{r}} \tag{15}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}. \tag{16}$$

What does each of the four terms in (16) mean? We can interpret $\ddot{r}$ as the purely "radial" acceleration and $r\ddot{\theta}$ as the purely "tangential" acceleration. You may remember that in circular motion, the centripetal acceleration is $r\dot{\theta}^2$ pointing radially inwards. It turns out that the last term $2\dot{r}\dot{\theta}$ represents the **Coriolis** force, which can be interpreted as a fictitious force in a rotating non-inertial frame. Wikipedia does a far better job illustrating its effects than I can hope to emulate[2].

To summarize the various terms of acceleration in the polar basis:

$$\ddot{\mathbf{r}} = \Big(\underbrace{\ddot{r}}_{\text{radial}}\ \overbrace{-\,r\dot{\theta}^2}^{\text{centripetal}}\Big)\,\hat{\mathbf{r}} + \Big(\underbrace{r\ddot{\theta}}_{\text{tangential}}\ \overbrace{+\,2\dot{r}\dot{\theta}}^{\text{Coriolis}}\Big)\,\hat{\boldsymbol{\theta}}. \tag{17}$$

[2] https://en.wikipedia.org/wiki/Coriolis_force

5 Preview of the Kepler Problem

Why do we bother writing acceleration in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$? It all depends on the kind of problems you want to solve. No one in their right mind would try to solve linear motion in the polar basis when the Cartesian one suffices.

For problems that have rotational symmetry, however, the polar basis is a natural choice. Later this week you will learn that angular momentum is always conserved in central force problems (that is, the force only has a radial component in $\hat{\mathbf{r}}$ but not $\hat{\boldsymbol{\theta}}$). Here we show a quick proof using the polar basis.

In a central force problem, the force vector can be written as

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}}. \tag{18}$$

By Newton's second law, the resulting acceleration on a point of mass m is

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} = \frac{f(r)}{m}\hat{\mathbf{r}}. \tag{19}$$

This means that the tangential component in (16) is zero:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{20}$$

Applying a trick[3] of multiplying both sides by r, we find that the left side can be written as a total time derivative:

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0, \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0. \tag{22}$$

We will learn that the angular momentum of a point mass in the 2D plane is simply $mr^2\dot{\theta}$, with m a constant. Equation (22) shows us that the total time derivative of this quantity is zero, that is, the quantity is constant over time. Thus, angular momentum is conserved[4].

The Kepler problem, which seeks the equations of motion of two bodies under mutual gravitation, is a central force problem with $f(r) \propto 1/r^2$. Therefore, the total angular momentum is also conserved in the Kepler problem. This is one of the key insights that allow us to solve the Kepler problem in the course of completing the OD project.

[3] `https://knowyourmeme.com/memes/im-gonna-do-whats-called-a-pro-gamer-move`

[4] It is not a coincidence that the force lacking a dependence on θ leads to the conservation of angular momentum. The profound truth underlying this connection is Noether's theorem, which roughly states that every continuous symmetry of a physical system leads to a conservation law. For example, symmetry in position leads to the conservation of momentum, and symmetry in time leads to the conservation of energy. Precisely stating and proving the theorem require the machinery of Lagrangian mechanics, which is beyond the scope of SSP.